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# Actions for Non-Abelian Twisted Self-Duality

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## Abstract

The dynamics of abelian vector and antisymmetric tensor gauge fields can be described in terms of twisted self-duality equations. These first-order equations relate the  $p$ -form fields to their dual forms by demanding that their respective field strengths are dual to each other. It is well known that such equations can be integrated to a local action that carries on equal footing the  $p$ -forms together with their duals and is manifestly duality invariant. Space-time covariance is no longer manifest but still present with a non-standard realization of space-time diffeomorphisms on the gauge fields. In this paper, we give a non-abelian generalization of this first-order action by gauging part of its global symmetries. The resulting field equations are non-abelian versions of the twisted self-duality equations. A key element in the construction is the introduction of proper couplings to higher-rank tensor fields. We discuss possible applications (to Yang-Mills and supergravity theories) and comment on the relation to previous no-go theorems.

## 1 Introduction

The dynamics of abelian vector and antisymmetric tensor gauge fields can be described in terms of so-called twisted self-duality equations. A prominent example are Maxwell's equations: Rather than expressing them in the standard form of second-order differential equations for a single gauge field, they may be written in terms of a gauge field and its magnetic dual by demanding that their respective field strengths are dual to each other. An analogous system of first-order equations can be formulated in general for abelian  $p$ -forms in  $D$ -dimensional space-time and their dual  $(D - p - 2)$ -forms. It takes the schematic form

$$*\mathcal{F} = \Omega \mathcal{M} \mathcal{F}, \tag{1.1}$$

where  $\mathcal{F}$  combines the abelian field strengths of the original  $p$ -forms and their magnetic duals, and  $\Omega\mathcal{M}$  denotes the ‘twist matrix’ which squares to the same multiple of the identity as the Hodge star ‘ $*$ ’ on the associated field strengths. References [1, 2] have coined the term of ‘twisted self-duality equations’. The system (1.1) persists in generalized form in the presence of scalar fields, fermions, and Chern-Simons-type couplings as they typically arise in supersymmetric theories. In particular, the twist matrix in the general case depends on the scalar fields of the theory.

The formulation of the  $p$ -form dynamics in terms of the first-order system (1.1) plays an important role in exposing the duality symmetries of the theory. Of particular interest is the case of ‘self-dual’  $p$ -forms in a  $(2p + 2)$ -dimensional space-time. In this case, the global duality symmetries typically mix the original  $p$ -forms with their magnetic duals. Thus they remain symmetries of the equations of motion and only a subset thereof can be implemented in a local way as symmetries of the standard second-order action. A prime example is the electric/magnetic duality of Maxwell’s equations. Similarly, in maximal four-dimensional supergravity [3], the 28 electric vector fields of the  $N = 8$  supermultiplet combine with their magnetic duals into the fundamental 56-dimensional representation of the duality group  $E_{7(7)}$ , of which only an  $SL(8)$  subgroup is realized as a symmetry of the action of [3]. For even  $p$ , the dynamics described by the system (1.1) may not even be integrable to a second-order action, as it happens for the chiral supergravities in 6 and 10 dimensions [4], [5].

It is known since the work of Henneaux-Teitelboim [6] and Schwarz-Sen [7] that the twisted self-duality equations (1.1) can be derived from a first-order action that carries on equal footing the  $p$ -forms together with their dual fields and is manifestly duality invariant, see [8, 9] for earlier work. The price to pay for duality invariance is the abandonment of manifest general coordinate invariance of the action. Although not manifest, the latter may be restored with a non-standard realization of space-time diffeomorphisms on the gauge fields. Alternatively, these theories have been obtained as the gauge-fixed versions of non-polynomial Lagrangians with manifest space-time symmetry [10, 11]. Recent applications of such actions have led to a Lagrangian for  $N = 8$  supergravity that possesses full  $E_{7(7)}$  invariance, allowing to examine the role of the full  $E_{7(7)}$  in perturbative quantization of the theory [12, 13]. Moreover, it has been advocated recently [14, 15] that the existence of such duality symmetric actions underlying (1.1) may suggest to grant a more prominent role to duality covariance than to space-time covariance. This in turn may go along with the realization of the hidden symmetries underlying maximal supergravity theories [16, 17].

In this paper, we present a non-abelian version of the twisted self-duality equations (1.1) and a first-order action from which it can be derived. The action is constructed as a deformation of the abelian actions of [6, 7] by gauging part of their global symmetries. We use the embedding tensor formalism of [18, 19, 20], which describes the possible gaugings in terms of a constant embedding tensor, subject to a number of algebraic

consistency constraints. As a result, we find that the consistent gaugings of the first-order action are constrained by precisely the same set of algebraic constraints as the gaugings in the standard second-order formulation. A key part in the construction is the introduction of couplings to higher-order  $p$ -forms and a new topological term in the gauged theory. In general, the global duality group is broken by the explicit choice of an embedding tensor, but acts as a symmetry within the full class of gaugings. I.e. the non-abelian Lagrangian remains formally invariant under the action of the duality group, if the embedding tensor is treated as a spurionic object transforming under this group.

For transparency, we restrict the discussion in this paper to the case of vector fields in four-dimensional space-time; in the last section we comment on the (straightforward) generalization to  $p$ -forms in  $D$  dimensions. The construction of a first-order action for non-abelian gauge fields in particular allows to revisit the recent discussion of possible gaugings of (subgroups of) electric/magnetic duality [21, 22].

The paper is organized as follows: in sections 2 and 3, we briefly review the general structure of the abelian twisted self-duality equations for vector gauge fields in four space-time dimensions, and the associated duality-invariant first-order action of [6, 7]. In section 4, we covariantize the abelian action upon gauging a subset of its global symmetry generators. The gauge group is characterized by a constant embedding tensor, subject to a set of algebraic constraints. A key part in the construction of the covariant action is the introduction of antisymmetric two-form tensor fields which turn out to be the on-shell duals of the scalar fields of the theory. The non-abelian action furthermore comprises a topological term, introduced in section 5. In section 6, we present the full first-order action and show that it gives rise to a non-abelian version of the twisted self-duality equations (1.1), part of which now relate the scalar fields and the two-form potentials. We furthermore show that although not manifest, the action is invariant under four-dimensional coordinate transformations, if the action of space-time diffeomorphisms on the vector and tensor gauge fields is properly modified by on-shell vanishing contributions. In section 7 we illustrate the construction by a simple example: pure Yang-Mills theory in absence of other matter couplings. The resulting first-order Lagrangian describes the Yang-Mills vector fields together with their duals. It gives rise to the Yang-Mills field equations and in the limit of vanishing gauge coupling constant consistently reduces to the abelian action of [6, 7]. Electric/magnetic duality is explicitly broken by the choice of the embedding tensor. We close in section 8 with a discussion of possible applications of these theories and comment on the relation to previous no-go theorems.

## 2 Twisted self-duality equations

We start from the general second-order Lagrangian for  $n$  ‘electric’ vector fields  $A_\mu{}^\Lambda$  with abelian field strengths in four space-time dimensions  $\mathcal{F}_{\mu\nu}{}^\Lambda = 2\partial_{[\mu}A_{\nu]}{}^\Lambda$

$$\mathcal{L}_0 = \frac{1}{4}e_4\mathcal{I}_{\Lambda\Sigma}\mathcal{F}_{\mu\nu}{}^\Lambda\mathcal{F}^{\mu\nu\Sigma} + \frac{1}{8}\mathcal{R}_{\Lambda\Sigma}\varepsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\mu\nu}{}^\Lambda\mathcal{F}_{\rho\sigma}{}^\Sigma. \quad (2.1)$$

The (possibly scalar dependent) symmetric matrices  $\mathcal{R}$  and  $\mathcal{I}$  play the role of generalized theta angles and coupling constants, respectively. We use a metric with signature  $(-, +, +, +)$ ,  $e_4 \equiv \sqrt{\det g}$  denotes the square root of its determinant, and we use the completely antisymmetric Levi-Civita tensor density  $\varepsilon_{0123} = 1$ . The Bianchi identities and the equations of motion derived from (2.1) take the form

$$\partial_{[\mu}\mathcal{F}_{\nu\rho]}{}^\Lambda = 0 = \partial_{[\mu}\mathcal{G}_{\nu\rho]\Lambda}, \quad (2.2)$$

respectively, where

$$\mathcal{G}_{\mu\nu\Lambda} \equiv -\varepsilon_{\mu\nu\rho\sigma}\frac{\partial\mathcal{L}_0}{\partial\mathcal{F}_{\rho\sigma}{}^\Lambda} = \mathcal{R}_{\Lambda\Gamma}\mathcal{F}_{\mu\nu}{}^\Gamma - \frac{1}{2}e_4\varepsilon_{\mu\nu\rho\sigma}\mathcal{I}_{\Lambda\Gamma}\mathcal{F}^{\rho\sigma\Gamma}. \quad (2.3)$$

The equations of motion thus take the form of integrability equations that (on a topologically trivial space-time that we shall assume in the following) allow for the definition of the dual magnetic vector fields as  $\mathcal{F}_{\mu\nu\Lambda} \equiv 2\partial_{[\mu}A_{\nu]\Lambda} \equiv \mathcal{G}_{\mu\nu\Lambda}$ . In order to set up an  $\mathrm{Sp}(2n, \mathbb{R})$  covariant notation we introduce  $2n$ -dimensional symplectic indices  $M, N, \dots$ , such that  $Z^M = (Z^\Lambda, Z_\Lambda)$ . The original equations of motion (2.2) can then be rewritten in the form of covariant twisted self-duality equations [3, 1, 2] for the symplectically covariant field strength  $\mathcal{F}^M \equiv (\mathcal{F}^\Lambda, \mathcal{F}_\Lambda)$ :

$$\mathcal{F}_{\mu\nu}{}^M = -\frac{1}{2}e_4\varepsilon_{\mu\nu\rho\sigma}\Omega^{MN}\mathcal{M}_{NK}(\phi^i)\mathcal{F}^{\rho\sigma K}, \quad (2.4)$$

with the positive definite symmetric matrix  $\mathcal{M}_{MN}$

$$\mathcal{M}(\phi^i) \equiv \begin{pmatrix} -\mathcal{I} - \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & \mathcal{R}\mathcal{I}^{-1} \\ \mathcal{I}^{-1}\mathcal{R} & -\mathcal{I}^{-1} \end{pmatrix}, \quad (2.5)$$

and the  $\mathrm{Sp}(2n, \mathbb{R})$  invariant skew-symmetric tensor  $\Omega_{MN}$ <sup>1</sup>

$$\Omega = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}. \quad (2.6)$$

By  $\phi^i$  we denote the scalar fields of the theory on which the matrix  $\mathcal{M}_{MN}$  may depend. Equation (2.4) constitutes the explicit four-dimensional form of the general twisted self-duality equations (1.1).

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<sup>1</sup>The conjugate matrix  $\Omega^{MN}$  is defined by  $\Omega^{MN}\Omega_{NP} = -\delta^M_P$ . In the following, we use the symplectic matrix  $\Omega^{MN}$  to raise and lower fundamental indices  $M, N, \dots$  using north-west south-east conventions:  $X^M = \Omega^{MN}X_N$ , etc..

The form of the twisted self-duality equations (2.4) allows to exhibit the properties of the global duality symmetries of the theory [23] : Consider the generators  $t_\alpha$  of infinitesimal symmetries of the remaining matter couplings with linear action on the vector fields according to

$$\delta_\alpha A_\mu^M = -(t_\alpha)_N^M A_\mu^N . \quad (2.7)$$

The twisted self-duality equations (2.4) are invariant under the action (2.7) if the action of  $t_\alpha$  on the scalar fields of the theory induces a linear transformation

$$\delta_\alpha \mathcal{M}_{MN}(\phi^i) = \delta_\alpha \phi^i \partial_i \mathcal{M}_{MN}(\phi) \stackrel{!}{=} 2(t_\alpha)_{(M}^K \mathcal{M}_{N)K}(\phi) , \quad (2.8)$$

of the matrix  $\mathcal{M}_{MN}(\phi)$  and moreover leaves the matrix  $\Omega_{MN}$  invariant:

$$(t_\alpha)_M^K \Omega_{KN} = (t_\alpha)_N^K \Omega_{KM} . \quad (2.9)$$

The latter condition implies that the symmetry group must be embedded into the symplectic group:  $G_{\text{duality}} \subset Sp(2n, \mathbb{R})$ . In the absence of scalar fields and thus for a constant matrix  $\mathcal{M}_{MN}$ , it follows directly from (2.8) that the duality symmetries must further be embedded into the compact subgroup  $U(n) \subset Sp(2n, \mathbb{R})$ . For later use we denote the algebra of generators  $t_\alpha$  of  $G_{\text{duality}}$  as

$$[t_\alpha, t_\beta] = f_{\alpha\beta}^\gamma t_\gamma , \quad (2.10)$$

with structure constants  $f_{\alpha\beta}^\gamma$ . Only the subset of triangular generators

$$(t_\alpha)_M^N = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} , \quad (2.11)$$

is realized in a local way as symmetries of the second-order action (2.1). The remaining generators can be realized in a non-local way [8, 14]. A manifest off-shell realization of the full duality group  $G_{\text{duality}}$  is achieved by passing to an equivalent first-order action, reviewed in the next section.

### 3 Action for abelian twisted self-duality

We first briefly review the first-order action for the abelian twisted self-duality equations (2.4) as originally constructed in [6, 7] and recently revisited in [14, 15]. In the notation we closely follow reference [12]. As a first step, we split the four-dimensional coordinates into  $\{x^\mu\} \rightarrow \{x^0, x^i\}$  and the metric as

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + h_{ij}N^iN^j & h_{ij}N^i \\ h_{ij}N^j & h_{ij} \end{pmatrix} ,$$

into the standard lapse and shift functions. The twisted self-duality equations (2.4) take the form

$$\mathcal{F}_{0i}{}^M - N^j \mathcal{F}_{ji}{}^M = -\frac{1}{2} e_3 \varepsilon_{ijk} N \Omega^{MN} \mathcal{M}_{NK} \mathcal{F}^{jkK}, \quad (3.1)$$

where all spatial indices are raised with the metric  $h^{ij}$ . With the definitions

$$\begin{aligned} \mathcal{E}_i{}^M &\equiv \mathcal{F}_{0i}{}^M - N^j \mathcal{F}_{ji}{}^M, \\ \mathcal{B}_i{}^M &\equiv -\frac{1}{2} e_3 \varepsilon_{ijk} N \Omega^{MN} \mathcal{M}_{NK} \mathcal{F}^{jkK}, \end{aligned} \quad (3.2)$$

equations (2.4) thus reduce to

$$\mathcal{E}_i{}^M = \mathcal{B}_i{}^M. \quad (3.3)$$

These first-order equations are obtained from variation of the action

$$\mathcal{L}_{\text{kin,ab}} = \frac{e_3}{2N} h^{ij} (\mathcal{E}_i{}^M - \mathcal{B}_i{}^M) \mathcal{M}_{MN} \mathcal{B}_j{}^N, \quad (3.4)$$

where  $e_3 = \sqrt{\det h}$  denotes the determinant of the three-dimensional vielbein. This action breaks the four-dimensional general coordinate covariance, but is manifestly invariant under abelian gauge transformations and under the action of the global duality symmetries (2.7), (2.8). Variation of the action (3.4) gives rise to

$$\begin{aligned} \delta \mathcal{L}_{\text{kin,ab}} &= \Omega_{MN} \varepsilon^{imn} \partial_m (\mathcal{E}_n{}^N - \mathcal{B}_n{}^N) \delta A_i{}^M + \frac{3}{4} \Omega_{MN} \varepsilon^{imn} \partial_{[0} \mathcal{F}_{mn]}{}^N \delta A_i{}^M \\ &\quad - \frac{1}{4} \Omega_{MN} \varepsilon^{imn} \partial_i \mathcal{F}_{mn}{}^N \delta A_0{}^M + \text{total derivatives}. \end{aligned} \quad (3.5)$$

Upon taking into account the Bianchi identities, this shows that the Lagrangian (3.4) depends on the components  $A_0{}^M$  only via a total derivative such that the manifestly gauge invariant action (3.4) can be replaced by an equivalent Lagrangian that only depends on the spatial components  $A_i{}^M$  of the vector fields. The action thus gives rise to the equations of motion

$$\varepsilon^{imn} \partial_m (\mathcal{E}_n{}^{\#N} - \mathcal{B}_n{}^N) = 0, \quad (3.6)$$

where  $\mathcal{E}_n{}^{\#N} \equiv \mathcal{E}_n{}^N + \partial_n A_0{}^M$  denotes the  $A_0{}^M$ -independent part of  $\mathcal{E}_n{}^N$ . Integration of (3.6) on a topologically trivial space-time then allows to define a function  $A_0{}^M$  such that the original equations of motion (3.3) are satisfied. Moreover, it may be shown [6, 7, 15] that although not manifest, the action (3.4) is invariant under four-dimensional coordinate reparametrization.

## 4 Non-abelian gauge theory and two-form potentials

In this section, we start generalizing the above construction to the non-abelian case. More precisely, we study the situation when a subgroup of the global symmetry group

$G_{\text{duality}}$  of the above theory is gauged. We use the embedding tensor formalism [18, 19, 20], in which the gauging is parametrized by a constant embedding tensor  $\Theta_M^\alpha$ , such that covariant derivatives are given by

$$D_\mu \equiv \partial_\mu - g A_\mu^M X_M \equiv \partial_\mu - g A_\mu^M \Theta_M^\alpha t_\alpha, \quad (4.1)$$

with coupling constant  $g$  and the gauge group generators  $X_M \equiv \Theta_M^\alpha t_\alpha$  defined as linear combinations of the generators of the global duality group (2.10). In four space-time dimensions, the embedding tensor  $\Theta_M^\alpha$  is constrained by the linear relations [20]

$$X_{(MNK)} = 0, \quad (4.2)$$

where we have defined the “generalized structure constants”

$$X_{MN}{}^K \equiv (X_M)_N{}^K = \Theta_M^\alpha (t_\alpha)_N{}^K, \quad (4.3)$$

as the gauge group generators evaluated in the vector field representation.<sup>2</sup> Typically, the constraint (4.2) can be solved by projecting out some of the irreducible components of the tensor  $\Theta_M^\alpha$  with respect to the global symmetry group  $G_{\text{duality}}$ . It is therefore referred to as a ‘representation constraint’.<sup>3</sup> Furthermore, the matrix  $\Theta_M^\alpha$  is subject to the bilinear constraints

$$f_{\alpha\beta}{}^\gamma \Theta_M^\alpha \Theta_N^\beta + (t_\alpha)_N{}^P \Theta_M^\alpha \Theta_P{}^\gamma = 0, \quad (4.4)$$

$$\Omega^{MN} \Theta_M^\alpha \Theta_N^\beta = 0, \quad (4.5)$$

of which the first corresponds to a generalized Jacobi identity and the second one insures locality of the gauging (i.e. the existence of a symplectic frame in which all magnetic charges vanish).

The proper definition of covariant field strengths requires the introduction of two-form potentials  $B_{\mu\nu\alpha}$  [25, 26, 20] transforming in the adjoint representation of the global duality group  $G_{\text{duality}}$ . Explicitly, these field strengths are given by

$$\mathcal{H}_{\mu\nu}{}^M \equiv 2\partial_{[\mu} A_{\nu]}{}^M + g(X_N)_P{}^M A_{[\mu}{}^N A_{\nu]}{}^P - \frac{1}{2}g\Theta^{M\alpha} B_{\mu\nu\alpha}, \quad (4.6)$$

with a Stückelberg-type coupling to the two-form potentials  $B_{\mu\nu\alpha}$ . Later on, the field equations will identify these two-form potentials as the duals of the scalar fields of

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<sup>2</sup>Note that unless they identically vanish, these generalized structure constants are never anti-symmetric in the first two indices, but satisfy  $X_{(MN)}{}^K = \frac{1}{2}\Omega^{KL}\Omega_{NP}X_{LM}{}^P$  due to the symplectic embedding (2.9) and the linear constraint (4.2).

<sup>3</sup>In supersymmetric theories, this constraint typically also ensures supersymmetry of the Lagrangian. In the  $N = 1$  theories, a non-vanishing component  $X_{(MNK)}$  is related to the anomaly structure of the theory [24].



the theory, c.f. equation (6.3) below. Their introduction thus does not correspond to adding new degrees of freedom. Local gauge transformations are given by

$$\begin{aligned}\delta_\Lambda A_\mu^M &= D_\mu \Lambda^M + \frac{1}{2} g \Theta^{M\alpha} \Lambda_{\mu\alpha} , \\ \delta_\Lambda B_{\mu\nu\alpha} &= 2D_{[\mu} \Lambda_{\nu]\alpha} - 2(t_\alpha)_{MN} \left( \Lambda^M \mathcal{H}_{\mu\nu}^N - A_{[\mu}^M \delta_\Lambda A_{\nu]}^N \right) ,\end{aligned}\quad (4.7)$$

with parameters  $\Lambda^M$  and  $\Lambda_{\mu\alpha}$ , under which the field strengths (4.6) transform covariantly as

$$\delta_\Lambda \mathcal{H}_{\mu\nu}^M = -g \Lambda^K X_{KN}^M \mathcal{H}_{\mu\nu}^N . \quad (4.8)$$

The presence of the two-form fields in (4.6) is crucial for the covariant transformation behavior. We finally mention that the covariant field strengths (4.6) satisfy the generalized Bianchi identities

$$D_{[\mu} \mathcal{H}_{\nu\rho]}^M = -\frac{1}{6} g \Theta^{M\alpha} \mathcal{H}_{\mu\nu\rho\alpha} , \quad (4.9)$$

with the covariant non-abelian field strength  $\mathcal{H}_{\mu\nu\rho\alpha}$  of the two-form tensor fields, given by

$$\mathcal{H}_{\mu\nu\rho\alpha} = 3D_{[\mu} B_{\nu\rho]\alpha} + 6t_{\alpha PQ} A_{[\mu}^P \left( \partial_\nu A_{\rho]}^Q + \frac{1}{3} g X_{RS}^Q A_\nu^R A_{\rho]}^S \right) + \dots \quad (4.10)$$

and satisfying in turn the Bianchi identities

$$D_{[\mu} \mathcal{H}_{\nu\rho\sigma]\alpha} = \frac{3}{2} (t_\alpha)_{MN} \mathcal{H}_{[\mu\nu}^M \mathcal{H}_{\rho\sigma]}^N + \dots \quad (4.11)$$

Here, the dots indicate terms that vanish after contraction with  $\Theta_M^\alpha$  and thus remain invisible in this theory.

Covariantization of the action (3.4) with respect to the local gauge transformations (4.7) is now straightforward: we redefine electric and magnetic fields by covariantizing the previous definitions (3.2) according to

$$\begin{aligned}\mathcal{E}_i^M &\equiv \mathcal{H}_{0i}^M - N^j \mathcal{H}_{ji}^M , \\ \mathcal{B}_i^M &\equiv -\frac{1}{2} e_3 \varepsilon_{ijk} N \Omega^{MN} \mathcal{M}_{NK} \mathcal{H}^{jkK} ,\end{aligned}\quad (4.12)$$

and define the kinetic Lagrangian by covariantizing (3.4) as

$$\mathcal{L}_{\text{kin}} = \frac{e_3}{2N} h^{ij} (\mathcal{E}_i^M - \mathcal{B}_i^M) \mathcal{M}_{MN} \mathcal{B}_j^N , \quad (4.13)$$

with the new covariant  $\mathcal{E}_i^M$ ,  $\mathcal{B}_i^M$ . In this form, the kinetic term is manifestly invariant under the non-abelian gauge transformations (4.7), since the fields  $\mathcal{E}_i^M$  and  $\mathcal{B}_i^M$  transform covariantly according to (4.8) and the matrix  $\mathcal{M}_{MN}$  transforms according to

(2.8). A short calculation shows that the general variation of the new covariant kinetic Lagrangian gives rise to

$$\begin{aligned}\delta\mathcal{L}_{\text{kin}} &= \Omega_{MN} \varepsilon^{imn} D_m (\mathcal{E}_n^N - \mathcal{B}_n^N) \delta A_i^M + \frac{1}{8} \varepsilon^{imn} g \Theta_M^\alpha \mathcal{H}_{0mn\alpha} \delta A_i^M \\ &\quad - \frac{1}{24} \varepsilon^{imn} g \Theta_M^\alpha \mathcal{H}_{imn\alpha} \delta A_0^M - \frac{1}{8} \varepsilon^{imn} g \Theta_M^\alpha (\mathcal{E}_i^M - 2\mathcal{B}_i^M) \Delta B_{mn\alpha} \\ &\quad + \frac{1}{8} \varepsilon^{imn} g \Theta_M^\alpha N^j \mathcal{H}_{ji}^M \Delta B_{mn\alpha} + \frac{1}{8} \varepsilon^{imn} g \Theta_M^\alpha \mathcal{H}_{mn}^M \Delta B_{0i\alpha} ,\end{aligned}\quad (4.14)$$

up to total derivatives. Here, we have used the Bianchi identities (4.9), and introduced the ‘covariant variations’  $\Delta B_{\mu\nu\alpha} \equiv \delta B_{\mu\nu\alpha} - 2(t_\alpha)_{MN} A_{[\mu}^M \delta A_{\nu]}^N$ , for a more compact notation. The variation (4.14) cannot yet be the final answer. E.g., as it stands, the field equations induced by this Lagrangian imply vanishing of the field strength  $\Theta_M^\alpha \mathcal{H}_{mn}^M$  in contrast to the desired dynamics. Also variation w.r.t.  $B_{mn\alpha}$  induces first-order field equations that contradict (3.3). To cure these inconsistencies, the theory needs to be amended by a topological term that we will give in the following.

## 5 The non-abelian topological term

Let us consider the following topological term in order of the gauge coupling constant

$$\mathcal{L}_{\text{top}} = \frac{1}{16} \varepsilon^{\mu\nu\rho\sigma} g \Theta_M^\alpha B_{\mu\nu\alpha} \mathcal{H}_{\rho\sigma}^M + \frac{1}{12} \varepsilon^{\mu\nu\rho\sigma} g X_{PQM} \partial_\mu A_\nu^M A_\rho^P A_\sigma^Q ,\quad (5.1)$$

that is manifestly four-dimensional space-time covariant and does not depend on the space-time metric nor on the scalar fields of the theory. Similar terms have appeared in [20] for general gaugings with magnetic charges and in [27] without the two-forms for gaugings of triangular subgroups of  $G_{\text{duality}}$ . Here, the structure of this term is considerably simpler due to the symplectic covariance of the formalism.<sup>4</sup> In particular, this topological term is separately gauge invariant. The general variation of (5.1) is given by

$$\delta\mathcal{L}_{\text{top}} = -\frac{1}{24} \varepsilon^{\mu\nu\rho\sigma} g \Theta_M^\alpha (\mathcal{H}_{\mu\nu\rho\alpha} \delta A_\sigma^M - \frac{3}{2} \mathcal{H}_{\mu\nu}^M \Delta B_{\rho\sigma\alpha}) ,\quad (5.2)$$

from which one readily deduces its invariance under local gauge transformations (4.7) according to

$$\delta_\Lambda \mathcal{L}_{\text{top}} = -\frac{3}{8} \varepsilon^{\mu\nu\rho\sigma} g \Lambda^K X_{(KMN)} \mathcal{H}_{\mu\nu}^M \mathcal{H}_{\rho\sigma}^N = 0 ,\quad (5.3)$$

with the Bianchi identities (4.9), (4.11) and the linear constraint (4.2).

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<sup>4</sup>Note in particular the absence of the  $A^4$  and the  $B^2$  terms in (5.1) and (5.2), which is due to the identities (4.2), (4.5) and their consequences such as

$$2X_{MQ[P} X_{RS]}^Q + 3X_{QM[P} X_{RS]}^Q = 0 ,$$

etc..

Upon the  $3+1$  split of the coordinates, the variation of the topological term nicely combines with the variation (4.14) of the kinetic term, and together they give rise to

$$\begin{aligned}\delta(\mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{top}}) &= \varepsilon^{imn} \left\{ \Omega_{MN} D_m (\mathcal{E}_n^M - \mathcal{B}_n^M) + \frac{1}{4} g \Theta_M^\alpha \mathcal{H}_{0mn\alpha} \right\} \delta A_i^M \\ &\quad - \frac{1}{12} \varepsilon^{imn} g \Theta_M^\alpha \mathcal{H}_{imn\alpha} \delta A_0^M \\ &\quad - \frac{1}{4} \varepsilon^{imn} g \Theta_M^\alpha (\mathcal{E}_i^M - \mathcal{B}_i^M) \Delta B_{mn\alpha} ,\end{aligned}\tag{5.4}$$

up to total derivatives. The relative factor between  $\mathcal{L}_{\text{kin}}$  and  $\mathcal{L}_{\text{top}}$  is chosen such that the combined action does not depend on the components  $B_{0i\alpha}$  while it depends on the components  $B_{ij\alpha}$  and  $A_0^M$  only upon projection with the embedding tensor via  $\Theta_M^\alpha B_{ij\alpha}$  and  $\Theta_M^\alpha A_0^M$ , respectively. This is the proper generalization of the  $A_0^M$ -independent action (3.4) in the abelian case. Let us also note, that in the full gauged theory, the field equations will receive further contributions from variation of the matter Lagrangian that is now coupled to the gauge fields:

$$\frac{\delta \mathcal{L}_{\text{matter}}}{\delta A_\mu^M} \equiv e_4 g j^\mu_M ,\tag{5.5}$$

with the covariant currents  $j^\mu_M$ . We will show in the next section, that the field equations induced by the variation (5.4), (5.5) give the proper non-abelian extension of the twisted self-duality equations.

## 6 Action and equations of motion

The full non-abelian Lagrangian thus is given by the sum of (4.13), (5.1) and possible matter couplings (scalars, fermions, gravity)

$$\begin{aligned}\mathcal{L}_{\text{covariant}} &= \frac{e_3}{2N} h^{ij} (\mathcal{E}_i^M - \mathcal{B}_i^M) \mathcal{M}_{MN} \mathcal{B}_j^N \\ &\quad + \frac{1}{16} \varepsilon^{\mu\nu\rho\sigma} g \Theta_M^\alpha B_{\mu\nu\alpha} \mathcal{H}_{\rho\sigma}^M + \frac{1}{12} \varepsilon^{\mu\nu\rho\sigma} g X_{PQM} \partial_\mu A_\nu^M A_\rho^P A_\sigma^Q \\ &\quad + \mathcal{L}_{\text{matter}} ,\end{aligned}\tag{6.1}$$

with the covariant electric and magnetic fields  $\mathcal{E}_i^M$ ,  $\mathcal{B}_i^M$  from (4.12). The variation of (6.1) is given in (5.4), (5.5). Formally, the Lagrangian (6.1) is invariant under the full action of the global duality group  $G_{\text{duality}}$ , if the embedding tensor  $\Theta_M^\alpha$  is treated as a spurionic object transforming under  $G_{\text{duality}}$ . A concrete choice of the embedding tensor will specify a particular gauging and explicitly break  $G_{\text{duality}}$ . The choice of the embedding tensor is restricted by the algebraic consistency conditions (4.2), (4.5). These coincide with the constraints imposed on the embedding tensor in the second-order formulation of the theory [20]. We will show now that the action (6.1) implies

the following set of equations of motion

$$\mathcal{E}_i^M = \mathcal{B}_i^M, \quad (6.2)$$

$$\Theta_M^\alpha \mathcal{H}_{\mu\nu\rho\alpha} = -2e_4 \varepsilon_{\mu\nu\rho\sigma} j^\sigma_M, \quad \text{with } j^\sigma_M \equiv e_4^{-1} g^{-1} \frac{\delta \mathcal{L}_{\text{matter}}}{\delta A_\sigma^M}, \quad (6.3)$$

of which the first is the non-abelian version of the twisted self-duality equation (3.3), and the second is the four-dimensional duality equation relating the two-form fields to the scalar fields or, more precisely, to the Noether current of the invariances that have been gauged. These equations parallel the equations of motion obtained from the second-order formalism with magnetic gauge fields of [20].

In order to derive the equations of motion (6.2), (6.3) we proceed in analogy with the abelian case of section 3 and make use of the fact that the full Lagrangian does not depend on the components  $B_{0i\alpha}$  and depends on the  $A_0^M$  only upon projection with the embedding tensor as  $\Theta_M^\alpha A_0^M$ . We then show that the equations of motion for the remaining fields precisely allow for the introduction of the missing  $B_{0i\alpha}$  and  $A_0^M$  such that equations (6.2), (6.3) are satisfied. To this end, we first choose a basis of vector fields  $A_\mu^M$  and of global symmetry generators  $t_\alpha$ , such that the embedding tensor  $\Theta_M^\alpha$  of the theory takes the block diagonal form

$$\Theta_M^\alpha = \begin{pmatrix} \Theta_A^a & \Theta_{\underline{A}}^a \\ \Theta_{\underline{A}}^a & \Theta_{\underline{A}}^a \end{pmatrix} \equiv \begin{pmatrix} \Theta_A^a & 0 \\ 0 & 0 \end{pmatrix}, \quad (6.4)$$

with invertible  $\Theta_A^a$ . I.e. only the gauge fields  $A_\mu^A$  participate in the gauging and they gauge the subgroup spanned by generators  $t_a$ . The quadratic constraint (4.5) implies that in this basis  $\Omega^{AB} = 0$ . In analogy to (3.6) we denote by  $\mathcal{E}_i^{\#M}$  the part of  $\mathcal{E}_i^M$  which is independent of  $B_{0i\alpha}$  and  $A_0^{\underline{A}}$ . We note that in the above basis  $\mathcal{E}_i^A = \mathcal{E}_i^{\#A}$ . Then, variation of (6.1) w.r.t.  $B_{ij\alpha}$  according to (5.4) gives rise to the equations

$$\mathcal{E}_n^A = \mathcal{B}_n^A, \quad (6.5)$$

Next, from variation w.r.t.  $A_i^{\underline{A}}$  we obtain

$$0 = \epsilon^{imn} \Omega_{\underline{AB}} D_m (\mathcal{E}_n^{\#B} - \mathcal{B}_n^B) = \epsilon^{imn} \Omega_{\underline{AB}} \partial_m (\mathcal{E}_n^{\#B} - \mathcal{B}_n^B), \quad (6.6)$$

where the second equality makes use of (4.2) and (6.5). From this equation we deduce (in analogy to (3.6)) the existence of a function  $A_0^B$  such that

$$\Omega_{\underline{AB}} (\mathcal{E}_n^{\#B} - \mathcal{B}_n^B) \equiv \Omega_{\underline{AB}} \partial_n A_0^B \implies \Omega_{\underline{AB}} \mathcal{E}_n^B = \Omega_{\underline{AB}} \mathcal{B}_n^B. \quad (6.7)$$

In general, the matrix  $\Omega_{\underline{AB}}$  is not invertible such that  $A_0^B$  is not uniquely defined by this equation. The ambiguity precisely corresponds to the freedom of a gauge transformation (4.7) with parameter  $\Lambda_{0a}$ . Finally, we may choose the component  $B_{0ia}$  such that

$$\Omega_{AA} \mathcal{E}_n^A = \Omega_{AA} \mathcal{B}_n^A, \quad (6.8)$$

with the  $A_0^B$  chosen in (6.7). Together, equations (6.5), (6.7), and (6.8), build the non-abelian twisted self-duality equation (6.2). The remaining parts of the  $\delta A_0^A$  and  $\delta A_i^A$  variations of (5.4), (5.5) combine into the second duality equation (6.3). We have thus shown, that the field equations induced by the covariantized Lagrangian (6.1) give rise to the set of non-abelian field equations (6.2), (6.3).

Let us finally discuss the symmetries of the action (6.1). Built as the sum of two gauge invariant terms, the action is clearly invariant under the local gauge transformations (4.7). Although four-dimensional coordinate invariance is not manifest, it follows from (5.4) after some calculation, that the action is also invariant under time-like diffeomorphisms  $\xi^0$ , provided the gauge field potentials transform as

$$\begin{aligned}\delta A_i^M &= \xi^0 (\mathcal{B}_i^M + N^j \mathcal{H}_{ji}^M) , \\ \delta A_0^M &= 0 , \\ \Theta_M^\alpha \Delta B_{ij\alpha} &= -2e_3 N \xi^0 \varepsilon_{ijk} j^k{}_M .\end{aligned}\tag{6.9}$$

On-shell, i.e. upon using the equations of motion (6.2), (6.3), these transformation laws reduce to

$$\delta A_i^M \approx \xi^0 \mathcal{H}_{0i} , \quad \Theta_M^\alpha \delta B_{ij\alpha} \approx \Theta_M^\alpha \xi^0 \mathcal{H}_{0ij\alpha} + 2(t_\alpha)_{MN} A_{[i}^M \delta A_{j]}^N ,\tag{6.10}$$

which in turn reproduce the standard transformation behavior under time-like diffeomorphisms up to local gauge transformations (4.7) with parameters  $\Lambda^M \equiv -\xi^0 A_0^M$ ,  $\Lambda_{\mu\alpha} \equiv -B_{\mu i\alpha} - (t_\alpha)_{KL} A_\mu^K A_0^L$ . In the abelian case, the modified form of time-like diffeomorphisms (6.9) for the vector fields  $A_i^M$  coincides with the results of [6, 7]. The main difference in the non-abelian case (apart from standard gauge covariantization) is the explicit appearance of the gauge fields in the matter part  $\mathcal{L}_{\text{matter}}$  of the Lagrangian. Since these couplings are manifestly four-dimensional coordinate covariant, the off-shell modification of the transformation law (6.9) leads to extra contributions from this sector which are proportional to the equations of motion

$$\delta_{\text{extra}} \mathcal{L} = \frac{\delta \mathcal{L}_{\text{matter}}}{\delta A_i^M} \xi^0 (\mathcal{B}_i^M - \mathcal{E}_i^M) .\tag{6.11}$$

With (5.4) and (5.5), these contributions are precisely cancelled by the modified transformation law of  $B_{ij}^M$  in (6.9). To summarize, the Lagrangian (6.1) is invariant under local gauge transformations (4.7) and also under four-dimensional diffeomorphisms with the modified transformation laws (6.9).

## 7 Example: Yang-Mills theory

The construction of a first-order action for non-abelian gauge theories presented above hinges on the solution of the algebraic consistency conditions (4.2), (4.5) for the embedding tensor  $\Theta_M^\alpha$ . We stress once more that these conditions are precisely the same

as in the second-order formulation of the theory [20]. In particular, the solutions of these equations encode all possible gaugings of a given abelian theory. The most interesting examples of such theories arise from extended gauged supergravities [28, 29], for which solutions to (4.2), (4.5) can be found by group-theoretical methods. E.g. the presented construction immediately gives rise to a first-order action for the maximal  $SO(8)$  gauged supergravity of [30], obtained as a deformation of the abelian first-order action of [12].

In this section, as an illustration of the construction we will discuss a much simpler (and somewhat degenerate) example: pure Yang-Mills theory in the absence of further matter content. The starting point for the construction of this theory is the abelian action (2.1) for  $n$  vector fields with constant  $\mathcal{I}$ , and  $\mathcal{R} = 0$ , corresponding to  $n$  copies of the standard Maxwell action. Accordingly, the matrix  $\mathcal{M}_{MN}$  is constant, such that the global symmetry of the twisted self-duality equations (2.4) and the first-order action (3.4) is the compact  $U(n)$ . Its  $U(1)$  subgroup is the standard electric/magnetic duality. Possible gaugings of this theory are described by an embedding tensor according to (4.1) which is a  $2n \times n^2$  matrix. A quick counting shows that the linear constraints (4.2) reduce the number of independent components in  $\Theta_M^\alpha$  to  $n^2(n-1)$ . In particular, pure Maxwell theory ( $n = 1$ ) does not allow a gauging of its duality symmetry. This is in accordance with the explicit recent results of [22]. For  $n = 2$ , the linear constraints allow for 4 independent parameters, however, it may be shown that the bilinear constraints (4.5) in this case do not admit any real solution. Non-trivial gaugings exist only for  $n > 2$  Maxwell fields. For  $n = 3$  one finds that the non-trivial solution of the consistency constraints is unique up to conjugation and corresponds to gauging an  $SO(3)$  subgroup of the global  $U(3)$ . This is a special case of the general construction given in the following.

Let us consider pure Yang-Mills theory with compact semi-simple gauge group  $G$  and  $n \equiv \dim G$  gauge fields  $A_\mu^A$  with the standard embedding  $G \subset SO(n) \subset U(n)$ . We denote by  $f_{AB}^C$  and  $\eta_{AB}$  the structure constants and the Cartan-Killing form of  $G$ , respectively. The duality covariant formulation of the theory will employ  $2n$  vector fields  $A_\mu^M = (A_\mu^A, C_{\mu A})$ , where for clarity we denote the dual vectors by  $C_{\mu A}$ . The embedding tensor is chosen such that the generalized structure constants (4.3) take the form

$$X_{MN}^K \equiv \begin{cases} X_{AB}^C &= -f_{AB}^C \\ X_A^B{}_C &= f_{AC}^B \end{cases} , \quad (7.1)$$

with all other components vanishing. It is straightforward to verify that they satisfy the symplectic embedding (2.9), and the linear constraint (4.2), while the bilinear constraints (4.5), reduce to the standard Jacobi identities for the structure constants  $f_{AB}^C$ . Moreover, this choice of the embedding tensor explicitly breaks the electric/magnetic duality  $U(1)$  (even though the gauge group commutes with it). The constant matrix

$\mathcal{M}_{MN}$  is parametrized by the Cartan-Killing form  $\eta_{AB}$  and its inverse  $\eta^{AB}$

$$\mathcal{M}_{MN} = \begin{pmatrix} \eta_{AB} & 0 \\ 0 & \eta^{AB} \end{pmatrix}. \quad (7.2)$$

The non-abelian twisted self-duality equations (6.2) in this case take the form

$$\mathcal{F}_{\mu\nu}{}^A = -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\eta^{AB}\mathcal{G}^{\rho\sigma}{}_B, \quad (7.3)$$

with the field strengths

$$\begin{aligned} \mathcal{F}_{\mu\nu}{}^A &\equiv 2\partial_{[\mu}A_{\nu]}{}^A - gf_{BC}{}^A A_\mu{}^B A_\nu{}^C, \\ \mathcal{G}_{\mu\nu}{}^A &= 2D_{[\mu}C_{\nu]}{}^A + gB_{\mu\nu}{}^A, \end{aligned} \quad (7.4)$$

where we have lowered and raised indices  $A, B, \dots$ , with  $\eta_{AB}$  and its inverse, respectively, and redefined the two-forms according to  $B_{\mu\nu}{}^A \equiv \frac{1}{2}\Theta_A{}^\alpha B_{\mu\nu\alpha} + f_{ABC} A_{[\mu}{}^B C_{\nu]}{}^C$  with respect to the previous notation. Gauge transformations (4.7) are given by

$$\begin{aligned} \delta A_\mu{}^A &= D_\mu \Lambda^A, & \delta C_\mu{}^A &= D_\mu \tilde{\Lambda}^A - g\Lambda_\mu{}^A, \\ \delta B_{\mu\nu}{}^A &= 2D_{[\mu}\Lambda_{\nu]}{}^A + gf_{BC}{}^A \left( \Lambda^B \mathcal{G}_{\mu\nu}{}^C - \tilde{\Lambda}^B \mathcal{F}_{\mu\nu}{}^C \right) + 2gf_{BC}{}^A C_{[\mu}{}^B \delta A_{\nu]}{}^C, \end{aligned} \quad (7.5)$$

with independent parameters  $\Lambda^A$ ,  $\tilde{\Lambda}^A$  and  $\Lambda_\mu{}^A$ . This illustrates explicitly how the two-form potential  $B_{\mu\nu}{}^A$  together with its gauge invariance cures the inconsistencies of a gauge theory with vector fields  $C_\mu{}^A$  that do not participate in the gauging but are charged under the gauge group.<sup>5</sup> The non-abelian field strength (4.10) of the two-forms takes the form

$$\mathcal{H}_{\mu\nu\rho}{}^A = 3D_{[\mu}B_{\nu\rho]}{}^A - 3gf_{ABC}\mathcal{F}_{[\mu\nu}{}^B C_{\rho]}{}^C, \quad (7.6)$$

and it vanishes on-shell according to its equations of motion (6.3). It is straightforward to verify that (7.3) together with the first-order equation  $\mathcal{H}_{\mu\nu\rho}{}^A = 0$  is equivalent to the original Yang-Mills field equations for the vector field  $A_\mu{}^A$ . The full Lagrangian (6.1) takes the explicit form

$$\begin{aligned} \mathcal{L}_{\text{covariant}} &= -\frac{1}{4}\mathcal{F}_{jk}{}^A\eta_{AB}\mathcal{F}^{jkB} - \frac{1}{4}\mathcal{G}_{jk}{}^A\eta_{AB}\mathcal{G}^{jkB} - \frac{1}{6}g\varepsilon^{ijk}\mathcal{H}_{ijk}{}^A A_0{}^A \\ &\quad + \frac{1}{4}\varepsilon^{ijk}\dot{C}_i{}^A\eta_{AB}\mathcal{F}_{jk}{}^B - \frac{1}{4}\varepsilon^{ijk}\dot{A}_i{}^A\eta_{AB}(\mathcal{G}_{jk}{}^B + gB_{jk}{}^B), \end{aligned} \quad (7.7)$$

which gives the first-order Lagrangian for pure Yang-Mills theory, describing the Yang-Mills vector fields  $A_\mu{}^M$  together with their duals  $C_\mu{}^A$ . However, this Lagrangian no

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<sup>5</sup>The explicit form of (7.5) shows that the construction in this example becomes somewhat degenerate: the gauge transformations  $\tilde{\Lambda}^A$  of the dual vector fields  $C_\mu{}^A$  may be entirely absorbed into a redefinition of the tensor gauge parameter  $\Lambda_\mu{}^A$ .

longer exhibits an analogue of the electric/magnetic  $U(1)$  symmetry, since the choice of the embedding tensor (7.1) explicitly breaks this  $U(1)$ , although the gauge group commutes with it. This is in accordance with the explicit no-go results of [8]. The field equations derived from (7.7) are equivalent to the Yang-Mills equations  $D^\mu \mathcal{F}_{\mu\nu}{}^A = 0$ , while in the limit  $g \rightarrow 0$ , the Lagrangian consistently reduces to the first-order duality invariant Lagrangian (3.4) describing a set of Maxwell equations. On the other hand, it follows directly, that by integrating out the two-form potentials  $B_{ij}{}^A$ , also the magnetic vector fields  $C_i{}^A$  disappear from (7.7), and the Lagrangian reduces to the original second-order Yang-Mills form

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4} \eta_{AB} \mathcal{F}_{\mu\nu}{}^A \mathcal{F}^{\mu\nu}{}^B, \quad (7.8)$$

which is manifestly four-dimensional covariant.

## 8 Conclusions

In this paper, we have constructed the general non-abelian deformations of the first-order actions of [6, 7], whose field equations give rise to a non-abelian version of the twisted self-duality equations (1.1). Consistent gaugings are characterized by the choice of a constant embedding tensor satisfying the set of algebraic constraints (4.2), (4.5). The resulting action (6.1) is formally invariant under the global duality group  $G_{\text{duality}}$ , if the embedding tensor itself transforms as a spurionic object under the duality group. The duality symmetry is broken when the embedding tensor is set to a particular constant value. Space-time covariance of the action is no longer manifest, but can be restored by modifying the action of diffeomorphisms on the gauge fields by additional contributions proportional to the equations of motion. The construction relies on the introduction of additional couplings to higher-order  $p$ -forms and a topological term.

The set of algebraic consistency constraints for the embedding tensor coincides with the one found for gaugings in the standard second-order formalism [20]. In other words, every gauging of the standard second-order action admits an analogous first-order action carrying all  $2n$  vector fields. We call to mind that in the second-order formulation typically only a part of the global duality group  $G_{\text{duality}}$  is realized as a symmetry of the ungauged action. Nevertheless, more general subgroups of  $G_{\text{duality}}$  can be gauged upon the introduction of magnetic vector fields. In the first-order formalism of this paper, all gaugings are directly obtained as gauging of part of the off-shell symmetries  $G_{\text{duality}}$  of the first-order action (3.4).

Despite the equivalence of the resulting field equations, the first-order and second-order formulations of gaugings remain complementary in several aspects: The second-order action [20] carries as many two-form potentials as the gauging involves magnetic vector fields. In particular, there always exists a symplectic frame in which all vector fields involved in the gauging are electric and no two-forms appear in the action.



In contrast, the first-order action (6.1) inevitably carries a fixed number of two-form potentials equal to the dimension of the gauge group. In the second-order formalism, space-time diffeomorphisms are realized in the standard way, while the canonical gauge transformations of the two-form potentials need to be modified by contributions proportional to the field equations [20]. Gauge transformations in the first-order formalism (4.7) in contrast are of the canonical form whereas it is the action of space-time diffeomorphisms that is modified by on-shell vanishing contributions. On the technical side of the construction, in the second-order formalism it is only the sum of kinetic and topological term of gauge fields that is gauge invariant. In the first-order formalism these terms are separately gauge invariant, but only a particular combination of them gives rise to a consistent set of field equations.

We emphasize that the construction of non-abelian gauge deformations of the first-order action (3.4) inevitably requires the introduction of two-form potentials with Stückelberg-type couplings to the vector fields (4.6) and a  $BF$ -type coupling in the topological term (5.1). It has recently been concluded in [21] that in absence of two-form potentials the first-order action (3.4) does not admit *any* non-abelian deformation, see [31] for earlier no-go results. Specifically, the deformations studied in [21] are triggered by antisymmetric structure constants  $C_{MN}^K \sim X_{[MN]}^K$ , corresponding to the ansatz (4.1), (4.3), with the additional assumption that  $X_{(MN)}^K \equiv 0$ . Indeed, in the general construction of non-abelian deformations, it is the symmetric part of the generalized structure constants  $X_{(MN)}^K$  that governs the introduction and additional couplings to two-form potentials [26]. Imposing the absence of two-form potentials thus requires that  $X_{(MN)}^K \equiv 0$ . However, in four dimensions this additional constraint is in direct contradiction to the constraints on the embedding tensor, cf. footnote 2, and implies that the  $X_{MN}^K$  vanish identically, in accordance with the analysis of [21]. Hence, there are no Yang-Mills-type deformations of the first-order action (3.4) in the absence of two-form potentials. In contrast, we have shown that by proper inclusion of such higher-order  $p$ -forms, the first-order action can be consistently generalized to the non-abelian case.

The construction of the first-order action (6.1) for non-abelian gauge fields furthermore allows to revisit the recent discussion of possible gaugings of (subgroups of) electric/magnetic duality [21, 22]. The answer to the question which subgroups of the global duality group  $G_{\text{duality}}$  can be gauged, now turns out to be encoded in the set of algebraic consistency constraints (4.2), (4.5) for the embedding tensor. Indeed, we have shown in section 7 that for pure Maxwell theory ( $n = 1$ , no scalar fields), these constraints do not admit any solution, thus reproducing the no-go result of [22]. In contrast, for a set of  $n$  Maxwell fields, we find that any compact group of dimension  $n$  can be gauged via the standard embedding  $G \subset SO(n) \subset U(n)$  into the global duality group. The resulting Lagrangian explicitly breaks electric/magnetic duality, in accordance with the analysis of [8]. In the presence of scalar fields, the global du-

ality group of (3.4) is enlarged to some non-compact subgroup of  $Sp(2n, \mathbb{R})$  and the possibilities for admissible gauge groups further increase, in particular due to the existence of different inequivalent symplectic frames. Many solutions to the corresponding consistency constraints have been found in the second-order formalism for extended gauged supergravities [28, 29]. E.g. the presented construction immediately gives rise to a deformation of the duality-invariant first-order action of [12], that describes the maximal  $SO(8)$  gauged supergravity of [30].

In this paper, we have restricted the discussion to vector fields in four space-time dimensions. Further coupling to fermions and the realization of supersymmetry will proceed straightforwardly along the lines of [12]. Moreover, the embedding tensor formalism employed in this paper serves as a guide as to how the presented construction can be straightforwardly generalized to describe non-abelian deformations of arbitrary  $p$ -forms in  $D$  space-time dimensions. The key structure underlying these constructions is the structure of non-abelian deformations of the associated hierarchy of  $p$ -form tensor fields [26, 32, 33]. In analogy to the theories constructed in this paper, the deformations of first-order actions of  $p$ -forms will generically involve couplings to higher-order  $(p+1)$ -forms and necessitate contributions from additional topological terms. While in the example of Yang-Mills theory discussed in the previous section, the introduction of additional two-form potentials may still occur as somewhat artificial, in higher-dimensional supergravity theories typically the full hierarchy of anti-symmetric  $p$ -form fields is already present in the original theory. It appears rather natural that all of them should be involved in the new couplings of the first-order action. Such a construction should culminate in a first-order description of the higher-dimensional theories in which all  $p$ -forms are present, related to their duals by the proper non-abelian extension of the actions of [6, 7, 14], and linked to their adjacent forms by Stückelberg-type couplings analogous to (4.6). This formulation may be particularly useful in the study of dimensional reductions and the realization of the hidden symmetries  $E_{10}$ ,  $E_{11}$ , [16, 17], extending the duality symmetries.

Another obvious avenue of generalization of the presented construction is the lift of the non-abelian deformations to the non-polynomial Lagrangians with manifest space-time symmetry [10, 11] from which the first-order actions [6, 7] are obtained after particular gauge fixing. Also the generalization to deformations of the first-order formulations based on a more general decomposition of space-time into  $d + (D - d)$  dimensions [34] (generalizing the case  $d = 1$  discussed in this paper) should be achievable along similar lines.

Finally, a particularly interesting realm of applications of the first-order actions (3.4) and their deformations described in this paper, are those chiral theories which do not admit the formulation of a standard second-order action. In this context, we mention most notably the six-dimensional chiral tensor gauge theories and recent attempts to construct their non-abelian deformations supposed to underlie the description of

multiple M5-brane dynamics [35, 36, 37, 38, 39]. The construction presented in this paper may provide a well-defined framework for the general study of such interactions.

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